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2009 J. Phys. A: Math. Theor. 42 025206

(http://iopscience.iop.org/1751-8121/42/2/025206)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 025206 (17pp) doi:10.1088/1751-8113/42/2/025206

Generalized hypergeometric photon-added and photon-depleted coherent states

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Received 9 July 2008, in final form 25 October 2008
Published 4 December 2008
Online at stacks.iop.org/JPhysA/42/025206

Abstract
In this paper, we introduce a family of photon-added as well as photon-depleted coherent states related to the inverse of ladder operators acting on hypergeometric coherent states. Their squeezing and antibunching properties are investigated in both conventional (nondeformed) and deformed quantum optics.

PACS numbers: 42.50.Ar, 42.50.Dv

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Coherent states are now of utmost importance in several areas of theoretical physics ranging from quantum optics to statistical mechanics and quantum field theory [1, 2]. These states which emerge from the study of the quantum harmonic oscillator were introduced in the early years of quantum mechanics [3] as wave packets whose dynamics resembles that of a classical particle in a quadratic potential. Different representations of the harmonic oscillator have been discussed extensively in the literature. The basic operators are the boson annihilation and creation operators a and $a^\dagger$, satisfying the usual commutation relation $[a, a^\dagger] = I$. These states are defined as the eigenstates of a.

The conventional coherent states have always been considered as the most classical ones (among the pure quantum states, of course). Moreover, they can serve as a starting point to introduce the nonclassical states that attracted considerable attention in quantum optics over the last two decades. Such states exhibit some purely quantum statistical properties such as squeezing, antibunching (sub-Poissonian statistics) (see [4] and references therein) and thus possess latent applications in optical communication and in precision and sensitive measurements [5].

To construct various families of nonclassical states, it is enough to make slight modifications in each definition of the conventional coherent states mentioned above. For these reasons many class of states, which are labeled nowadays as nonclassical, appeared in the literature as some kinds of generalized coherent states. To cite a few, let us mention the following:

(i) The binomial states, defined as finite-linear superposition of field number states $|n\rangle$ weighted by a binomial counting probability distribution

$$|K;p;\phi\rangle = \sum_{n=0}^{K}\left[\binom{K}{n}p^n(1-p)^{K-n}\right]^{1/2}\mathrm{e}^{\mathrm{i}n\phi}|n\rangle. \tag{1}$$

(ii) The coherent states defined from the deformations of the canonical commutation relations. Among these, one can mention the so-called maths-type and physics-type coherent states and the coherent states à la Quesne (see [6] and references therein).

(iii) The nonlinear coherent states [7–9], which are defined as the right-hand eigenstates of the product of the boson annihilation operator a and a non-constant function of the number operator $N = a^\dagger a$,

$$f(N)a|\alpha, f\rangle = \alpha|\alpha, f\rangle \tag{2}$$

where $f(N)$ is an operator-valued function of the number operator and α is a complex eigenvalue.

Recently, Appl *et al* [10] have constructed and studied generalized hypergeometric coherent states. These states are defined such that their normalization functions are given in terms of generalized hypergeometric functions. They are defined by

$$\begin{aligned}|p;q;z\rangle &\equiv |a_1,\ldots,a_p;b_1,\ldots,b_q;z\rangle_{(p;q)}\\ &= {}_p\mathcal{N}_q^{-1/2}(|z|^2)\sum_{n=0}^{+\infty}\frac{z^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle\end{aligned} \tag{3}$$

with the strictly positive parameter functions (of the discrete variable n)

$$\begin{aligned}{}_p\rho_q(n) &\equiv {}_p\rho_q(a_1,\ldots,a_p;b_1,\ldots,b_q;n)\\ &= \Gamma(n+1)\frac{(b_1)_n\ldots(b_q)_n}{(a_1)_n\ldots(b_p)_n}\end{aligned} \tag{4}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. The generalized hypergeometric (coherent) states (GH(C)Ss) are eigenstates of the lowering operator

$${}_pU_q \equiv \sum_{n=0}^{+\infty}{}_pf_q(n)|n\rangle\langle n+1| \tag{5}$$

with

$${}_pf_q(n) = \sqrt{(n+1)\frac{(n+b_1)\ldots(n+b_q)}{(n+a_1)\ldots(n+a_p)}}, \tag{6}$$

i.e. ${}_pU_q|p;q;z\rangle = z|p;q;z\rangle$. The action of the operators ${}_pU_q$ and its adjoint ${}_pU_q^\dagger$ on the Fock basis $\{|n\rangle, n = 0,1,2,\ldots\}$ is given by

$${}_pU_q|n\rangle = {}_pf_q(n-1)|n-1\rangle \tag{7a}$$

$${}_pU_q^\dagger|n\rangle = {}_pf_q(n)|n+1\rangle. \tag{7b}$$

The aim of this paper is to construct and discuss the quantum statistical properties of the photon-added and photon-depleted states corresponding to the generalized hypergeometric coherent states $|p; q; z\rangle$. These states are constructed from the inverse of ladder operators acting on the states $|p; q; z\rangle$. The states built in this way have extensively been studied in the past decade as they can considerably reduce noise in any signoise [11] and can be produced in nonlinear processes in cavities [12]. In the following, if there are no numerator (denominator), this will be indicated by a dash, for example $|0; 1; z\rangle = |-; b; z\rangle_{(0;1)}$.

This paper is organized as follows. In section 2, we introduce the generalized inverse of the raising and lowering operators by their actions on the Fock space and discuss some of their properties. In sections 3 and 4, respectively, we introduce the photon-added and photon-depleted states corresponding to the generalized hypergeometric coherent states $|p; q; z\rangle$ as the eigenstates of the combination of the ladder operators ${}_pU_q$ and ${}_pU_q^\dagger$ and their inverse operators. The physical properties of these states in the context of conventional quantum optics are studied in section 5. Since ${}_pU_q$ and ${}_pU_q^\dagger$ generalize the conventional annihilation and creation operators, we re-examine the physical properties of those states for deformed photons, described by the operators ${}_pU_q$ and ${}_pU_q^\dagger$ in the context of deformed states in section 6. Finally, conclusions are drawn in section 7.

2. Generalized inverse of boson operators

By following the work of Metha *et al* [13], we define the generalized inverse of boson operators, ${}_pU_q^{-1}$ and $\left({}_pU_q^\dagger\right)^{-1}$ in terms of their actions on the number states $|n\rangle$ as follows:

$$ {}_pU_q^{-1}|n\rangle = \frac{1}{{}_pf_q(n)}|n+1\rangle \tag{8a} $$

$$ \left({}_pU_q^\dagger\right)^{-1}|n\rangle = \frac{1}{{}_pf_q(n-1)}(1-\delta_{n,0})|n-1\rangle. \tag{8b} $$

The relation

$$ {}_pU_q\,{}_pU_q^{-1}|n\rangle = {}_pU_q\left(\frac{1}{{}_pf_q(n)}|n+1\rangle\right) = |n\rangle \tag{9} $$

shows that ${}_pU_q^{-1}$ is the right inverse of ${}_pU_q$ while $\left({}_pU_q^\dagger\right)^{-1}$ is the left inverse of ${}_pU_q^\dagger$, i.e.,

$$ {}_pU_q\,{}_pU_q^{-1} = \left({}_pU_q^\dagger\right)^{-1}{}_pU_q^\dagger = I. \tag{10} $$

We further note that ${}_pU_q^{-1}\,{}_pU_q$ and ${}_pU_q^\dagger\left({}_pU_q^\dagger\right)^{-1}$ are given by

$$ {}_pU_q^{-1}\,{}_pU_q = {}_pU_q^\dagger\left({}_pU_q^\dagger\right)^{-1} = I - |0\rangle\langle 0| \tag{11} $$

where $|0\rangle\langle 0|$ is the projection operator in the vacuum. Indeed, if $n \neq 0$, ${}_pU_q^{-1}\,{}_pU_q|n\rangle = {}_pf_q(n-1)\left({}_pU_q^{-1}|n-1\rangle\right) = |n\rangle$ and ${}_pU_q^{-1}\,{}_pU_q|0\rangle = 0$, i.e., ${}_pU_q^{-1}\,{}_pU_q|n\rangle = (I - |0\rangle\langle 0|)|n\rangle$.

Therefore, ${}_pU_q^{-1}$ behaves as a creation operator of a photon, while $\left({}_pU_q^\dagger\right)^{-1}$ behaves as an annihilation operator of a photon.

Proposition 2.1. *Let ${}_pU_q^{-m}$ (resp. $\left({}_pU_q^\dagger\right)^{-m}$) be the mth power of ${}_pU_q^{-1}$ (resp. of $\left({}_pU_q^\dagger\right)^{-1}$). Then, the following relations hold:*

$$ {}_pU_q^{m}\,{}_pU_q^{-m} = \left({}_pU_q^\dagger\right)^{-m}\left({}_pU_q^\dagger\right)^{m} = I \tag{12a} $$

$$ {}_pU_q^{-m}\,{}_pU_q^{m} = \left({}_pU_q^\dagger\right)^{m}\left({}_pU_q^\dagger\right)^{-m} = I - \sum_{j=0}^{m-1}|j\rangle\langle j|. \tag{12b} $$

Proof. (12*a*) can be readily checked. (12*b*) can be recursively proved. Indeed, we suppose that it holds for $j < m-1$. For $j = m$, we have

$$\begin{aligned}
{}_pU_q^{-m}\,{}_pU_q^{m} &= {}_pU_q^{-(m-1)}\,{}_pU_q^{-1}\,{}_pU_q^{1}\,{}_pU_q^{m-1} = {}_pU_q^{-(m-1)}(I - |0\rangle\langle 0|)\,{}_pU_q^{m-1} \\
&= I - \sum_{j=0}^{m-2} |j\rangle\langle j| - {}_pU_q^{-(m-1)}|0\rangle\langle 0|\,{}_pU_q^{m-1} \\
&= I - \sum_{j=0}^{m-2} |j\rangle\langle j| - |m-1\rangle\langle m-1| = I - \sum_{j=0}^{m-1} |j\rangle\langle j|. \qquad (13)
\end{aligned}$$

□

Since the GHS $|p; q; z\rangle$ is an eigenstate of the lowering operator ${}_pU_q$, it would seem that it is also an eigenstate of ${}_pU_q^{-1}$ with eigenvalue z^{-1}. However, it is not so. In fact,

$$\begin{aligned}
{}_pU_q^{-1}|p; q; z\rangle &= {}_p\mathcal{N}_q^{-1/2}(|z|^2)\sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{{}_p\rho_q(n)}}\,{}_pU_q^{-1}|n\rangle \\
&= {}_p\mathcal{N}_q^{-1/2}(|z|^2)\sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{{}_p\rho_q(n)}}\frac{1}{{}_pf_q(n)}|n+1\rangle \\
&= {}_p\mathcal{N}_q^{-1/2}(|z|^2)\sum_{n=1}^{+\infty} \frac{z^{n-1}}{\sqrt{{}_p\rho_q(n)}}|n\rangle \\
&= {}_p\mathcal{N}_q^{-1/2}(|z|^2)z^{-1}\left(\sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle - |0\rangle\right) \\
&= z^{-1}\left(|p; q; z\rangle - {}_p\mathcal{N}_q^{-1/2}(|z|^2)|0\rangle\right) \qquad (14)
\end{aligned}$$

where use has been made of the relation ${}_pf_q(n)\sqrt{{}_p\rho_q(n)} = \sqrt{{}_p\rho_q(n+1)}$. This means that the action of the operator ${}_pU_q^{-1}$ on the GHS $|p; q; z\rangle$ yields a one-photon excitation state in GHS.

Similarly, the GHS $|p; q; z\rangle$ is not an eigenstate of $\left({}_pU_q^{\dagger}\right)^{-1}$, though $\left({}_pU_q^{\dagger}\right)^{-1}$ is a lowering operator:

$$\left({}_pU_q^{\dagger}\right)^{-1}|p; q; z\rangle = z\,{}_p\mathcal{N}_q^{-1}(|z|^2)\sum_{n=0}^{+\infty} z^n \frac{\sqrt{{}_p\rho_q(n)}}{{}_p\rho_q(n+1)}|n\rangle. \qquad (15)$$

Hence, there exists no right eigenstate of the operator $\left({}_pU_q^{\dagger}\right)^{-1}$ except for the vacuum state. So, the action of the operator $\left({}_pU_q^{\dagger}\right)^{-1}$ on $|p; q; z\rangle$ yields a one-photon annihilation state in GHSs. Therefore, we have

Proposition 2.2. *The repeated action of* ${}_pU_q^{-1}$ *and* $\left({}_pU_q^{\dagger}\right)^{-1}$ *on the GHSs* $|p; q; z\rangle$ *yields the multiphoton-excitation states*

$$\begin{aligned}
|p; q; z, +m\rangle &= {}_p\mathcal{C}_q^{-1/2}(|z|^2; m)\,{}_pU_q^{-m}|p; q; z\rangle \\
&= {}_p\mathcal{C}_q^{-1/2}(|z|^2; m)z^{-m}\left(|p; q; z\rangle - {}_p\mathcal{N}_q^{-1/2}(|z|^2)\sum_{n=0}^{m-1} \frac{z^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle\right) \qquad (16)
\end{aligned}$$

with the normalization function

$${}_p\mathcal{C}_q(|z|^2; m) = |z|^{-2m}\left(1 - {}_p\mathcal{N}_q^{-1}(|z|^2)\sum_{n=0}^{m-1} \frac{|z|^{2n}}{{}_p\rho_q(n)}\right)$$

and the multiphoton-annihilation states

$$|p;q;z,-m\rangle = {}_p\mathcal{S}_q^{-1/2}(|z|^2;m)\big({}_pU_q^\dagger\big)^{-m}|p;q;z\rangle$$
$$= {}_p\mathcal{S}_q^{-1/2}(|z|^2;m)\,{}_p\mathcal{N}_q^{-1/2}(|z|^2)z^m\sum_{n=0}^{+\infty} z^n\frac{\sqrt{{}_p\rho_q(n)}}{{}_p\rho_q(n+m)}|n\rangle \tag{17}$$

with the normalization function

$${}_p\mathcal{S}_q(|z|^2;m) = |z|^{2m}\,{}_p\mathcal{N}_q^{-1}(|z|^2)\sum_{n=0}^{+\infty}|z|^{2n}\frac{{}_p\rho_q(n)}{({}_p\rho_q(n+m))^2}$$

where m is a positive integer.

Proof. From the relations

$$({}_pU_q)^{-m}|n\rangle = \frac{1}{{}_pf_q(n)\cdots{}_pf_q(n+m-1)}|n+m\rangle$$
$$= \frac{\sqrt{{}_p\rho_q(n)}}{\sqrt{{}_p\rho_q(n+m)}}|n+m\rangle \tag{18}$$

and

$$\big({}_pU_q^\dagger\big)^{-m}|n\rangle = \frac{(1-\delta_{n,0})(1-\delta_{n-1,0})\cdots(1-\delta_{n-m+1,0})}{{}_pf_q(n-1)\,{}_pf_q(n-2)\cdots{}_pf_q(n-m)}|n-m\rangle$$
$$= \frac{\sqrt{{}_p\rho_q(n-m)}}{\sqrt{{}_p\rho_q(n)}}(1-\delta_{n,0})(1-\delta_{n-1,0})\cdots(1-\delta_{n-m+1,0})|n-m\rangle \tag{19}$$

we readily obtain

$${}_pU_q^{-m}|p;q;z\rangle = {}_p\mathcal{N}_q^{-1/2}(|z|^2)\sum_{n=m}^{+\infty}\frac{z^{n-m}}{\sqrt{{}_p\rho_q(n)}}|n\rangle \tag{20}$$

$$\big({}_pU_q^\dagger\big)^{-m}|p;q;z\rangle = {}_p\mathcal{N}_q^{-1/2}(|z|^2)\sum_{n=m}^{+\infty}\frac{z^n\sqrt{{}_p\rho_q(n-m)}}{{}_p\rho_q(n)}|n-m\rangle \tag{21}$$

which can be used to get (16) and (17). □

The convergence of the normalization functions ${}_p\mathcal{C}_q(|z|^2;m)$ and ${}_p\mathcal{S}_q(|z|^2;m)$ does not depend on m. Indeed, these functions converge, in the following cases:

$$\text{for any } z \qquad \text{if} \quad p<q+1 \tag{22a}$$
$$|z|<1 \qquad \text{if} \quad p=q+1 \tag{22b}$$
$$|z|=1 \qquad \text{if} \quad p=q+1,\quad \eta<1 \tag{22c}$$
$$|z|=1,\quad z\neq 1 \qquad \text{if} \quad p=q+1,\quad 0\leqslant\eta\leqslant 1, \tag{22d}$$

where

$$\eta = \mathrm{Re}\left(\sum_{j=1}^{p}a_j - \sum_{j=1}^{q}b_j\right).$$

In all other cases they diverge [14].

3. Generalized hypergeometric photon-added coherent states

From the relations ${}_pU_q^m\,{}_pU_q^{-m} = I$ and ${}_pU_q|p;q;z\rangle = z|p;q;z\rangle$, it is easy to see that $|p;q;z,+m\rangle$ are the right eigenstates of the operators ${}_pU_q^{-m}\,{}_pU_q^{m+1}$ with eigenstates z, i.e.,

$$ {}_pU_q^{-m}\,{}_pU_q^{m+1}|p;q;z,+m\rangle = z|p;q;z,+m\rangle. \tag{23} $$

As the number states $\{|j\rangle, j = 0, 1, 2, \ldots, m-1\}$ are absent in the family of photon-added states $|p;q;z,+m\rangle$, the states $|p;q;z,+m\rangle$ cannot therefore form a complete set. However, each set of them, along with the number states $\{|j\rangle, j = 0, 1, 2, \ldots, m-1\}$, does form a complete set as we will see in the following.

Another family of generalized hypergeometric photon-added states can be constructed by repeating ${}_pU_q^\dagger$ on the states $|p;q;z\rangle$, namely

$$ \begin{aligned} |p;q;z,+m\rangle' &= {}_p\mathcal{C}'^{-1/2}_q(|z|^2;m)\left({}_pU_q^\dagger\right)^m|p;q;z\rangle \\ &= {}_p\mathcal{C}'^{-1/2}_q(|z|^2;m)\,{}_p\mathcal{N}_q^{-1/2}(|z|^2)\sum_{n=0}^{+\infty} z^n \frac{\sqrt{{}_p\rho_q(n+m)}}{{}_p\rho_q(n)}|n+m\rangle \end{aligned} \tag{24} $$

with the normalization function

$$ {}_p\mathcal{C}'_q(|z|^2;m) = {}_p\mathcal{N}_q^{-1}(|z|^2)\sum_{n=0}^{+\infty}|z|^{2n}\frac{{}_p\rho_q(n+m)}{({}_p\rho_q(n))^2}. \tag{25} $$

It can be easily proved that

$$ \left({}_pU_q^\dagger\right)^m {}_pU_q\left({}_pU_q^\dagger\right)^{-m}|p;q;z,+m\rangle' = z|p;q;z,+m\rangle'. \tag{26} $$

The relations $\left({}_pU_q^\dagger\right)^m {}_pU_q\left({}_pU_q^\dagger\right)^{-m}|j\rangle = 0$, $j \in \{0, 1, 2, \ldots, m\}$ prove that the number states $|0\rangle, |1\rangle, \ldots, |m\rangle$ are the eigenstates of the operators $\left({}_pU_q^\dagger\right)^m {}_pU_q\left({}_pU_q^\dagger\right)^{-m}$ with eigenvalue zero. Thus, we get $(m+1)$-fold degeneracy for this eigenvalue. Clearly, the states $|p;q;z,+m\rangle'$ does not form a complete set.

By recalling that, for $p < q+1$ or $p = q+1, \eta > 1$, the GHSs satisfy the resolution of unity [10]

$$ \frac{1}{\pi}\int \mathrm{d}^2z\,{}_pw_q(|z|^2)|p;q;z\rangle\langle p;q;z| = I \equiv \sum_{n=0}^{+\infty}|n\rangle\langle n| \tag{27} $$

with

$$ \begin{aligned} {}_pw_q(x) &= {}_pw_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) \\ &= \frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)}\,{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) \\ &\quad\times G^{q-1,0}_{p,q+1}(a_1-1,\ldots,a_p-1;b_1-1,\ldots,b_q-1;x) \end{aligned} \tag{28} $$

where ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x)$ is the generalized hypergeometric function and G is the Meijer function, we obtain the corresponding completeness relations of the family of the states $|p;q;z,+m\rangle$.

Proposition 3.1. *The states* $|p;q;z,+m\rangle$*, along with the number states* $\{|j\rangle, j = 0, 1, 2, \ldots, m-1\}$*, satisfy the completeness relation*

$$ \frac{1}{\pi}\int \mathrm{d}^2z\,{}_pw_q(|z|^2)\,{}_p\mathcal{C}_q(|z|^2;m)|p;q;z,+m\rangle\langle p;q;z,+m|\left({}_pU_q^\dagger\right)^m {}_pU_q^m + \sum_{j=0}^{m-1}|j\rangle\langle j| = I. \tag{29} $$

Proof. Indeed, by multiplying (27) by ${}_pU_q^{-m}$ on the left-hand side and by ${}_pU_q^{m}$ on the right-hand side, we obtain

$$\frac{1}{\pi}\int d^2z\, {}_pw_q(|z|^2)\, {}_pU_q^{-m}|p;q;z\rangle\langle p;q;z|\, {}_pU_q^{m} = I - \sum_{j=0}^{m-1}|j\rangle\langle j|. \tag{30}$$

By multiplying again on the right-hand side by $\left({}_pU_q^\dagger\right)^{-m}\left({}_pU_q^\dagger\right)^{m} = I$ and using the relations

$$\langle p;q;z,+m| = {}_p\mathcal{C}_q^{-1/2}(|z|^2;m)\langle p;q;z|\left({}_pU_q^\dagger\right)^{-m}$$

and

$${}_pU_q^{m}\left({}_pU_q^\dagger\right)^{-m}\left({}_pU_q^\dagger\right)^{m} = \left({}_pU_q^\dagger\right)^{-m}\left({}_pU_q^\dagger\right)^{m}{}_pU_q^{m}$$

we finally obtain

$$\frac{1}{\pi}\int d^2z\, {}_pw_q(|z|^2)\, {}_p\mathcal{C}_q(|z|^2;m)|p;q;z,+m\rangle\langle p;q;z,+m|\left({}_pU_q^\dagger\right)^{m}{}_pU_q^{m} + \sum_{j=0}^{m-1}|j\rangle\langle j| = I. \tag{31}$$

This equation shows that the states $|p;q;z,+m\rangle$, along with the number states $\{|j\rangle, j = 0,1,2,\ldots,m-1\}$, span a complete Hilbert space. □

Proposition 3.2. *The eigenstates of the operator* $\left({}_pU_q^\dagger\right)^{m}{}_pU_q\left({}_pU_q^\dagger\right)^{-m}$*, along with the number states* $\{|j\rangle, j = 0,1,2,\ldots,m-1\}$*, satisfy the completeness relation*

$$\frac{1}{\pi}\int d^2z\, {}_pw_q(|z|^2)\, {}_p\mathcal{C}'_q(|z|^2;m)|p;q;z,+m\rangle''\langle p;q;z,+m|\, {}_pU_q^{-m}\left({}_pU_q^\dagger\right)^{-m} + \sum_{j=0}^{m-1}|j\rangle\langle j| = I. \tag{32}$$

Proof. By first multiplying (27) by $\left({}_pU_q^\dagger\right)^{m}$ on the left-hand side and by ${}_pU_q^{m}$ on the right-hand side, we obtain

$$\frac{1}{\pi}\int d^2z\, {}_pw_q(|z|^2)\left({}_pU_q^\dagger\right)^{m}|p;q;z\rangle\langle p;q;z|\, {}_pU_q^{m} = \left({}_pU_q^\dagger\right)^{m}{}_pU_q^{m} \tag{33}$$

and then by ${}_pU_q^{-m}\left({}_pU_q^\dagger\right)^{-m}$, we get

$$\frac{1}{\pi}\int d^2z\, {}_pw_q(|z|^2)\, {}_p\mathcal{C}'_q(|z|^2;m)|p;q;z,+m\rangle''\langle p;q;z,+m|\, {}_pU_q^{-m}\left({}_pU_q^\dagger\right)^{-m} + \sum_{j=0}^{m-1}|j\rangle\langle j| = I \tag{34}$$

which proves the completeness of the eigenstates of the operator $\left({}_pU_q^\dagger\right)^{m}{}_pU_q\left({}_pU_q^\dagger\right)^{-m}$ along with the number states $\{|j\rangle, j = 0,1,2,\ldots,m-1\}$. □

4. Generalized hypergeometric photon-depleted coherent states

One can easily prove that $|p;q;z,-m\rangle$ are the right eigenstates of the operators $\left({}_pU_q^\dagger\right)^{-m}{}_pU_q\left({}_pU_q^\dagger\right)^{m}$ with eigenvalue z, i.e.

$$\left({}_pU_q^\dagger\right)^{-m}{}_pU_q\left({}_pU_q^\dagger\right)^{m}|p;q;z,-m\rangle = z|p;q;z,-m\rangle. \tag{35}$$

Besides, unlike the hypergeometric photon-added coherent states $|p;q;z,+m\rangle$, the family $|p;q;z,-m\rangle$ forms a complete set. The corresponding completeness relation reads

$$\frac{1}{\pi}\int d^2z\, {}_pw_q(|z|^2)\, {}_p\mathcal{S}_q(|z|^2;m)|p;q;z,-m\rangle\langle p;q;z,-m|\, {}_pU_q^{m}\left({}_pU_q^\dagger\right)^{m} = I. \tag{36}$$

Therefore, the states $|p;q;z,-m\rangle$ can be used as a basis of a complete representation.

5. Physical properties of the states $|p;q;z,+m\rangle$ and $|p;q;z,-m\rangle$ in nondeformed quantum optics

In this section, we proceed to study some physical properties of the states $|p;q;z,+m\rangle$ and $|p;q;z,-m\rangle$. For such a purpose, we need to evaluate the expectation values of some monomials in the boson creation and annihilation operator $a^\dagger, a$. These are defined by

$$\langle (a^\dagger)^{p'} a^r\rangle_{(+m)} = \langle p;q;z,+m|(a^\dagger)^{p'}a^r|p;q;z,+m\rangle \tag{37a}$$

$$\langle (a^\dagger)^{p'} a^r\rangle_{(-m)} = \langle p;q;z,-m|(a^\dagger)^{p'}a^r|p;q;z,-m\rangle. \tag{37b}$$

More explicitly, from the relations

$$a^r|n\rangle = \sqrt{\frac{n!}{(n-r)!}}|n-r\rangle \qquad 0\leqslant r\leqslant n \tag{38a}$$

$$(a^\dagger)^{p'}|n\rangle = \sqrt{\frac{(n+p')!}{n!}}|n+p'\rangle \tag{38b}$$

we obtain

$$\begin{aligned}\langle (a^\dagger)^{p'} a^r\rangle_{(+m)} &= {}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2;m)\sum_{j=0}^{+\infty}\sum_{n=\max(r-m,0)}^{+\infty}\frac{z^{*j}z^n}{\sqrt{{}_p\rho_q(n+m)}\sqrt{{}_p\rho_q(j+m)}}\\ &\quad\times\frac{\sqrt{(n+m)!(n+m-r+p')!}}{(n+m-r)!}\delta_{j,n+p'-r}\\ &= {}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2;m)z^{r-p'}\sum_{j=0}^{+\infty}\frac{|z|^{2j}}{\sqrt{{}_p\rho_q(j+m+r-p')}\sqrt{{}_p\rho_q(j+m)}}\\ &\quad\times\frac{\sqrt{(j+m+r-p')!(j+m)!}}{(j+m-p')!}\end{aligned} \tag{39}$$

and

$$\begin{aligned}\langle (a^\dagger)^{p'} a^r\rangle_{(-m)} &= {}_p\tilde{\mathcal{S}}_q^{-1}(|z|^2;m)z^{r-p'}\sum_{j=0}^{+\infty}\frac{|z|^{2(j+m)}\sqrt{{}_p\rho_q(j+r-p')}\sqrt{{}_p\rho_q(j)}}{{}_p\rho_q(j+m)\,{}_p\rho_q(j+m+r-p')}\\ &\quad\times\frac{\sqrt{(j+r-p')!j!}}{(j-p')!}\end{aligned} \tag{40}$$

where

$${}_p\tilde{\mathcal{S}}_q = {}_p\mathcal{S}_q\,{}_p\mathcal{N}_q, \qquad {}_p\tilde{\mathcal{C}}_q = {}_p\mathcal{C}_q\,{}_p\mathcal{N}_q. \tag{41}$$

5.1. Quadrature squeezing

Here, we study the quadrature squeezing of the states $|p;q;z,+m\rangle$ and $|p;q;z,-m\rangle$. For that, let us consider the following Hermitian quadrature operators:

$$X_1 = \frac{a+a^\dagger}{\sqrt{2}} \qquad X_2 = \frac{a-a^\dagger}{\sqrt{2}\mathrm{i}}. \tag{42}$$

Then, they satisfy the following uncertainty relation:

$$\langle\Delta X_1^2\rangle\langle\Delta X_2^2\rangle \geqslant \tfrac{1}{4} \tag{43}$$

where $\langle \Delta X^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2$. From (43), it follows that a state is squeezed if and only if any of the following conditions hold:

$$\langle \Delta X_1^2 \rangle < \tfrac{1}{2} \qquad \langle \Delta X_2^2 \rangle < \tfrac{1}{2}. \tag{44}$$

For the states $|p; q; z, \pm m\rangle$ these conditions can be expressed as

$$F_{jp} = 2\left(\langle X_j^2 \rangle_{(+m)} - (\langle X_j \rangle_{(+m)})^2\right) - 1 < 0 \qquad j = 1, 2 \tag{45}$$

$$F_{jn} = 2\left(\langle X_j^2 \rangle_{(-m)} - (\langle X_j \rangle_{(-m)})^2\right) - 1 < 0 \qquad j = 1, 2. \tag{46}$$

Now using the above results and the identities

$$2\langle \Delta X_1^2 \rangle - 1 = \langle a^2 \rangle + \langle a^2 \rangle^\dagger + 2\langle a^\dagger a \rangle - 2\langle a \rangle \langle a \rangle^\dagger - \langle a \rangle^2 - (\langle a \rangle^2)^\dagger \tag{47a}$$

and

$$2\langle \Delta X_2^2 \rangle - 1 = -\langle a^2 \rangle - \langle a^2 \rangle^\dagger + 2\langle a^\dagger a \rangle - 2\langle a \rangle \langle a \rangle^\dagger + \langle a \rangle^2 + (\langle a \rangle^2)^\dagger \tag{47b}$$

we find that squeezing in the quadrature X_1 occurs for the states $|p; q; z, +m\rangle$ whenever

$$\begin{aligned} F_{1p} = &\left({}_p\tilde{C}_q(|z|^2; m)\, {}_p\Omega_q(|z|^2; m) - {}_p\Lambda_q^2(|z|^2; m)\right) \cos(2\theta) \\ &+ {}_p\tilde{C}_q(|z|^2; m)\, {}_p\Upsilon_q(|z|^2; m) - {}_p\Lambda_q^2(|z|^2; m) < 0 \end{aligned} \tag{48}$$

and that in the quadrature X_2 occurs whenever

$$\begin{aligned} F_{2p} = &\left({}_p\Lambda_q^2(|z|^2; m) - {}_p\tilde{C}_q(|z|^2; m)\, {}_p\Omega_q(|z|^2; m)\right) \cos(2\theta) \\ &+ {}_p\tilde{C}_q(|z|^2; m)\, {}_p\Upsilon_q(|z|^2; m) - {}_p\Lambda_q^2(|z|^2; m) < 0 \end{aligned} \tag{49}$$

where

$${}_p\Lambda_q(|z|^2; m) = \sum_{n=0}^{+\infty} \frac{|z|^{2n}\sqrt{n+m+1}}{\sqrt{{}_p\rho_q(n+m)}\sqrt{{}_p\rho_q(n+m+1)}} \tag{50a}$$

$${}_p\Upsilon_q(|z|^2; m) = \sum_{n=0}^{+\infty} \frac{|z|^{2(n-1)}(n+m)}{{}_p\rho_q(n+m)} \tag{50b}$$

$${}_p\Omega_q(|z|^2; m) = \sum_{n=0}^{+\infty} \frac{|z|^{2n}\sqrt{(n+m+1)(n+m+2)}}{\sqrt{{}_p\rho_q(n+m)}\sqrt{{}_p\rho_q(n+m+2)}} \tag{50c}$$

$${}_p\tilde{C}_q(|z|^2; m) = \sum_{n=0}^{+\infty} \frac{|z|^{2n}}{{}_p\rho_q(n+m)} \tag{50d}$$

and the angle θ defined by $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$ is limited to the interval from 0 to π or from $-\pi/2$ to $\pi/2$. On the other hand, we find that the states $|p; q; z, -m\rangle$ exhibit squeezing in X_1 provided

$$\begin{aligned} F_{1n} = &\left({}_p\tilde{S}'_q(|z|^2; m)\, {}_pD_q(|z|^2; m) - {}_pA_q^2(|z|^2; m)\right) \cos(2\theta) \\ &+ {}_p\tilde{S}'_q(|z|^2; m)\, {}_pB_q(|z|^2; m) - {}_pA_q^2(|z|^2; m) < 0 \end{aligned} \tag{51}$$

where

$${}_pA_q(|z|^2; m) = \sum_{n=0}^{+\infty} \frac{|z|^{2n}\sqrt{{}_p\rho_q(n)}\sqrt{{}_p\rho_q(n+1)}\sqrt{n+1}}{{}_p\rho_q(n+m)\, {}_p\rho_q(n+m+1)} \tag{52a}$$

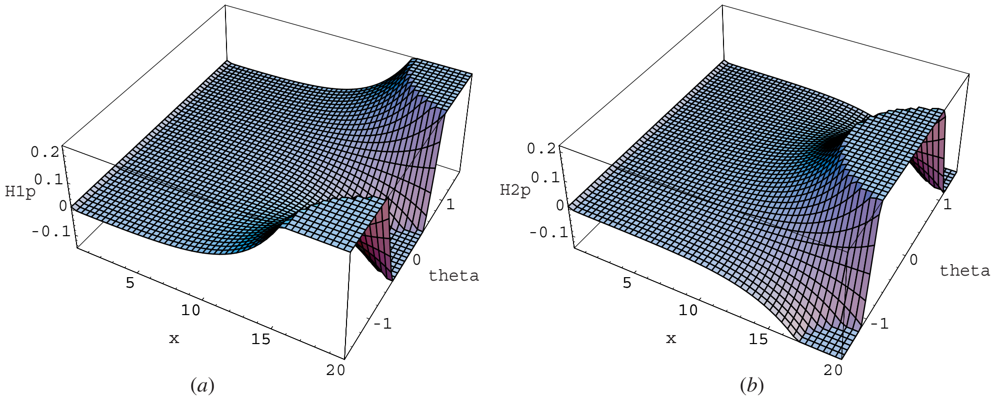


Figure 1. (*a*) Three-dimensional plot of F_{1p}, corresponding to squeezing of X_1 for the states $|-; b; z, +m\rangle$ versus $x = |z|^2$ and θ for $b = 1/30, m = 1$. (*b*) Three-dimensional plot of F_{2p}, corresponding to squeezing of X_2 for the states $|-; b; z, +m\rangle$ versus $x = |z|^2$ and θ for $b = 1/30$, $m = 1$.

$$ {}_pB_q(|z|^2; m) = \sum_{n=0}^{+\infty} \frac{|z|^{2(n-1)}\,{}_p\rho_q(n)n}{({}_p\rho_q(n+m))^2} \tag{52b} $$

$$ {}_pD_q(|z|^2; m) = \sum_{n=0}^{+\infty} \frac{|z|^{2n}\sqrt{(n+1)(n+2)}\sqrt{{}_p\rho_q(n)}\sqrt{{}_p\rho_q(n+2)}}{{}_p\rho_q(n+m)\,{}_p\rho_q(n+m+2)} \tag{52c} $$

$$ {}_p\tilde{S}'_q(|z|^2; m) = \sum_{n=0}^{+\infty} \frac{|z|^{2n}\,{}_p\rho_q(n)}{({}_p\rho_q(n+m))^2} \tag{52d} $$

and in the X_2 quadrature provided

$$ \begin{aligned} F_{2n} = {} & \left({}_pA_q^2(|z|^2; m) - {}_p\tilde{S}'_q(|z|^2; m)\,{}_pD_q(|z|^2; m)\right)\cos(2\theta) \\ & + {}_p\tilde{S}'_q(|z|^2; m)\,{}_pB_q(|z|^2; m) - {}_pA_q^2(|z|^2; m) < 0. \end{aligned} \tag{53} $$

In figures 1(*a*) and (*b*), we have shown the three-dimensional plot of $H_{1p} = 10^{-6}F_{1p}$ (where F_{1p} corresponding to squeezing of X_1 for the states $|-; b; z, +m\rangle$) and $H_{2p} = 10^{-6}F_{2p}$ (where F_{2p} corresponding to squeezing of X_2 for the states $|-; b; z, +m\rangle$) versus $x = |z|^2$ and θ, for $b = 1/30$ and $m = 1$. It is seen that the maximum squeezing of X_1 occurs at $\theta = \pm\pi/2$ while the maximum squeezing of X_2 occurs at $\theta = 0$. It is also observed that for $\theta = 0$, the variance $\langle \Delta X_1^2\rangle_{(+m)}$ is squeezed while $\langle \Delta X_2^2\rangle_{(+m)}$ turns to be unsqueezed as x increases.

In figures 2(*a*) and (*b*), respectively, the three-dimensional plots of $H_{1n} \equiv 10^{-6}F_{1n}$ (where F_{1n} corresponds to squeezing of X_1 for the states $|-; b; z, -m\rangle$) and $H_{2n} \equiv 10^{-6}F_{2n}$ (where F_{2n} corresponds to squeezing of X_2 for the states $|-; b; z, -m\rangle$) versus $x = |z|^2$ and θ for $b = 1/30$ and $m = 1$ are shown. We note that for larger x the maximum squeezing of X_1 occurs at $\theta = 0$ while the maximum squeezing of X_2 occurs at $\theta = \pm\pi/2$. It is also observed that for $\theta = 0$, in the states $|-; b; z, -m\rangle$, the quadrature X_1 is unsqueezed while X_2 is squeezed.

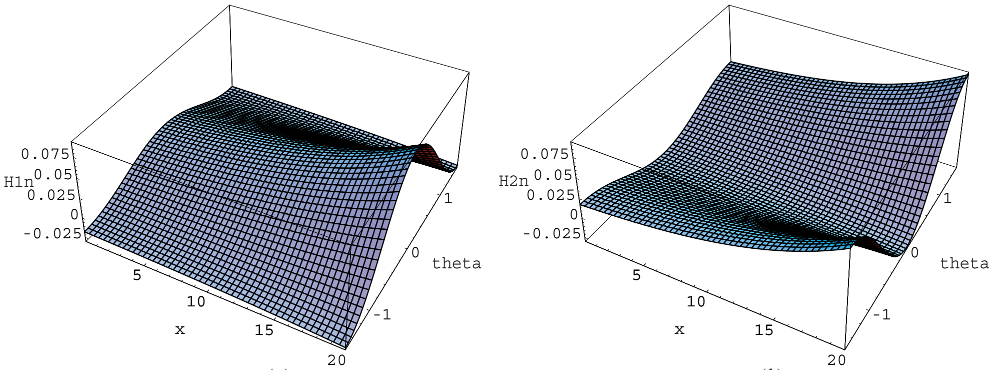


Figure 2. (a) Three-dimensional plot of $H_{1n} = 10^{-6}F_{1n}$, corresponding to squeezing of X_1 for the states $|-; b; z, -m\rangle$ versus $x = |z|^2$ and θ for $b = 1/30, m = 1$. (b) Three-dimensional plot of $H_{2n} \equiv 10^{-6}F_{2n}$, corresponding to squeezing of X_2 for the states $|-; b; z, -m\rangle$ versus $x = |z|^2$ and θ for $b = 1/30, m = 1$.

5.2. Photon number distribution

To examine sub-Poissonian behavior of the states $|p; q; z, +m\rangle$ and $|p; q; z, -m\rangle$, we need to consider the second-order correlation function, named the Mandel Parameter [15]

$$Q_{(\pm m)} = \frac{\langle (a^\dagger)^2 a^2\rangle_{(\pm m)} - \langle a^\dagger a\rangle^2_{(\pm m)}}{\langle a^\dagger a\rangle_{(\pm m)}}. \qquad (54)$$

Then, the states $|p; q; z, \pm m\rangle$ exhibit super-Poissonian/Poissonian/sub-Poissonian behavior according to $Q_{(\pm m)} > / = / < 1$.

By using (39) and (40) the Mandel parameter reads

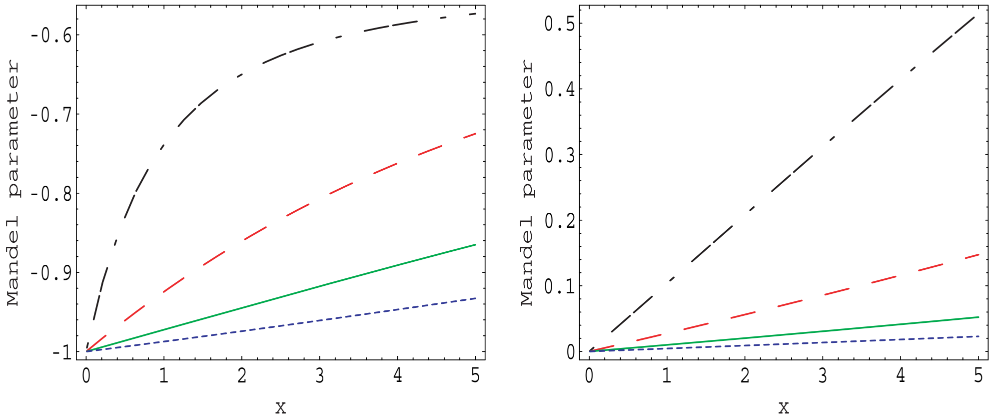
$$Q_{(+m)} = \left(\sum_{n=0}^{+\infty} \frac{|z|^{2n}(n+m)(n+m-1)}{{}_p\rho_q(n+m)}\right)\left(\sum_{n=0}^{+\infty} \frac{|z|^{2n}(n+m)}{{}_p\rho_q(n+m)}\right)^{-1} - \left(\sum_{n=0}^{+\infty} \frac{|z|^{2n}(n+m)}{{}_p\rho_q(n+m)}\right)\left(\sum_{n=0}^{+\infty} \frac{|z|^{2n}}{{}_p\rho_q(n+m)}\right)^{-1}, \qquad (55)$$

for the states $|p; q; z, +m\rangle$, and

$$Q_{(-m)} = \left(\sum_{n=0}^{+\infty} \frac{|z|^{2n}n(n-1)\,{}_p\rho_q(n)}{({}_p\rho_q(n+m))^2}\right)\left(\sum_{n=0}^{+\infty} \frac{|z|^{2n}n\,{}_p\rho_q(n)}{({}_p\rho_q(n+m))^2}\right)^{-1} - \left(\sum_{n=0}^{+\infty} \frac{|z|^{2n}n\,{}_p\rho_q(n)}{({}_p\rho_q(n+m))^2}\right)\left(\sum_{n=0}^{+\infty} \frac{|z|^{2n}\,{}_p\rho_q(n)}{({}_p\rho_q(n+m))^2}\right)^{-1}, \qquad (56)$$

for the states $|p; q; z, -m\rangle$.

In figures 3(a) and (b), we show the effects of the parameters a, b and m on the variation of the Mandel parameter as a function of x for the states $|-; b; z, +m\rangle$ and $|-; b; z, -m\rangle$, respectively. We conclude from these numerical results that the states $|-; b; z, +m\rangle$ always exhibit sub-Poissonian statistics, while by increasing x the states $|-; b; z, -m\rangle$ show a super-Poissonian behavior.



(*a*) (*b*)

Figure 3. (*a*) Variation of the Mandel parameter Q for the states $|-; b; z, +m\rangle$ versus $x = |z|^2$: $b = \frac{1}{30}, m = 1$(dotdash); $b = \frac{1}{30}, m = 2$(dash); $b = \frac{1}{30}, m = 3$(solid); $b = \frac{1}{30}, m = 4$(dot). (*b*) Variation of the Mandel parameter Q for the states $|-; b; z, -m\rangle$ versus $x = |z|^2$:$b = \frac{1}{30}$, $m = 1$(dotdash); $b = \frac{1}{30}, m = 2$(dash); $b = \frac{1}{30}, m = 3$(solid); $b = \frac{1}{30}, m = 4$(dot).

6. Physical properties in deformed quantum optics

For $p < q + 1$, the operators ${}_pU_q$ and ${}_pU_q^\dagger$ may be considered as a generalization of the usual annihilation and creation operators. Indeed, under this condition

$$ {}_0U_0 = \sum_{n=0}^{+\infty} \sqrt{n+1}|n\rangle\langle n+1| \equiv a, \qquad {}_0U_0^\dagger = \sum_{n=0}^{+\infty} \sqrt{n+1}|n+1\rangle\langle n| \equiv a^\dagger $$

and $|0; 0; z\rangle$ corresponds to the conventional coherent states. Hence, they may be interpreted as describing 'dressed' photons, which may be invoked in the phenomenological model explaining some observable phenomena [16]. The physical properties considered in the previous section may therefore be re-examined for the deformed photons.

To proceed with such an analysis, we first calculate the expectation values of various operators, namely, ${}_pU_q$, ${}_pU_q^2$, ${}_pU_q^\dagger {}_pU_q$, ${}_pU_q {}_pU_q^\dagger$ and $\left({}_pU_q^\dagger\right)^2 {}_pU_q^2$ in the states $|p; q; z, +m\rangle$ and $|p; q; z, -m\rangle$.

For the states $|p; q; z, +m\rangle$, one readily obtains

$$ \langle {}_pU_q\rangle_{(+m)} = {}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2; m) z \sum_{j=0}^{+\infty} \frac{|z|^{2j}}{{}_p\rho_q(j+m)} = z\,{}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2; m)\,{}_p\mathcal{O}_q(|z|^2; m) \tag{57a} $$

$$ \left\langle {}_pU_q^2\right\rangle_{(+m)} = {}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2; m) z^2 \sum_{j=0}^{+\infty} \frac{|z|^{2j}}{{}_p\rho_q(j+m)} = z^2\,{}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2; m)\,{}_p\mathcal{O}_q(|z|^2; m) \tag{57b} $$

$$ \left\langle {}_pU_q^\dagger {}_pU_q\right\rangle_{(+m)} = {}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2; m) \sum_{j=0}^{+\infty} \frac{|z|^{2j}}{{}_p\rho_q(j+m-1)} = {}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2; m)\,{}_p\mathcal{Q}_q(|z|^2; m) \tag{57c} $$

$$ \left\langle {}_pU_q {}_pU_q^\dagger\right\rangle_{(+m)} = {}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2; m) \sum_{j=0}^{+\infty} \frac{|z|^{2j}\,{}_pf_q^2(j+m)}{{}_p\rho_q(j+m)} = {}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2; m)\,{}_p\mathcal{Q}'_q(|z|^2; m) \tag{57d} $$

$$\left\langle \left({}_pU_q^\dagger\right)^2 {}_pU_q^2\right\rangle_{(+m)} = {}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2;m)\sum_{j=0}^{+\infty}\frac{|z|^{2j}}{{}_p\rho_q(j+m-2)} \tag{57e}$$

and for the states $|p;q;z,-m\rangle$, we have

$$\langle {}_pU_q\rangle_{(-m)} = {}_p\tilde{\mathcal{S}}_q^{-1}(|z|^2;m)z\sum_{j=0}^{+\infty}\frac{|z|^{2(j+m)}{}_p\rho_q(j+1)}{{}_p\rho_q(j+m){}_p\rho_q(j+m+1)} \tag{58a}$$

$$\left\langle {}_pU_q^2\right\rangle_{(-m)} = {}_p\tilde{\mathcal{S}}_q^{-1}(|z|^2;m)z^2\sum_{j=0}^{+\infty}\frac{|z|^{2(j+m)}{}_p\rho_q(j+2)}{{}_p\rho_q(j+m){}_p\rho_q(j+m+2)} \tag{58b}$$

$$\left\langle {}_pU_q^\dagger {}_pU_q\right\rangle_{(-m)} = {}_p\tilde{\mathcal{S}}_q^{-1}(|z|^2;m)\sum_{j=1}^{+\infty}\frac{|z|^{2(j+m)}{}_p\rho_q(j){}_pf_q^2(j-1)}{({}_p\rho_q(j+m))^2} \tag{58c}$$

$$\left\langle {}_pU_q {}_pU_q^\dagger\right\rangle_{(-m)} = {}_p\tilde{\mathcal{S}}_q^{-1}(|z|^2;m)\sum_{j=0}^{+\infty}\frac{|z|^{2(j+m)}{}_p\rho_q(j+1)}{({}_p\rho_q(j+m))^2} \tag{58d}$$

$$\left\langle \left({}_pU_q^\dagger\right)^2({}_pU_q)^2\right\rangle_{(-m)} = {}_p\tilde{\mathcal{S}}_q^{-1}(|z|^2;m)\sum_{j=2}^{+\infty}\frac{|z|^{2(j+m)}{}_p\rho_q(j){}_pf_q^2(j-1){}_pf_q^2(j-2)}{({}_p\rho_q(j+m))^2}. \tag{58e}$$

6.1. Quadrature squeezing in deformed quantum optics

To investigate the quantum fluctuations of the quadrature operators in the context of deformed quantum optics, we consider deformed quadratures

$$X_{1u} = \frac{{}_pU_q + {}_pU_q^\dagger}{2} \qquad \text{and} \qquad X_{2u} = \frac{{}_pU_q - {}_pU_q^\dagger}{2i}. \tag{59}$$

In any state, they satisfy the uncertainty relation

$$\left\langle \Delta X_{1u}^2\right\rangle\left\langle \Delta X_{2u}^2\right\rangle \geqslant \tfrac{1}{4}\,|\langle[X_{1u},X_{2u}]\rangle|^2. \tag{60}$$

Therefore, a state will exhibit amplitude squeezing if

$$\left\langle \Delta X_{1u}^2\right\rangle < \tfrac{1}{2}\,|\langle[X_{1u},X_{2u}]\rangle|, \qquad \left\langle \Delta X_{2u}^2\right\rangle < \tfrac{1}{2}\,|\langle[X_{1u},X_{2u}]\rangle| \tag{61}$$

which is equivalent to

$$F'_{ip} = \left\langle X_{iu}^2\right\rangle_{(+m)} - \langle X_{iu}\rangle_{(+m)}^2 - \tfrac{1}{4}\left|\left\langle\left[{}_pU_q,{}_pU_q^\dagger\right]\right\rangle_{(+m)}\right| < 0 \qquad i = 1,2 \tag{62}$$

$$F'_{in} = \left\langle X_{iu}^2\right\rangle_{(-m)} - \langle X_{iu}\rangle_{(-m)}^2 - \tfrac{1}{4}\left|\left\langle\left[{}_pU_q,{}_pU_q^\dagger\right]\right\rangle_{(-m)}\right| < 0 \qquad i = 1,2. \tag{63}$$

Therefore, by using (57*a*)–(57*d*) we find that

$$\begin{aligned} F'_{1p} = F'_{2p} &= \frac{1}{2}{}_p\tilde{\mathcal{C}}_q^{-1}(|z|^2;m)\left({}_p\mathcal{Q}_q(|z|^2;m) - |z|^2{}_p\tilde{\mathcal{C}}_q(|z|^2;m)\right) \\ &= \frac{1}{2}\frac{{}_p\mathcal{C}_q^{-1}(|z|^2;m)}{{}_p\rho_q(m-1)} \end{aligned} \tag{64}$$

if $m \geqslant 1$ and $F'_{1p} = F'_{2p} = 0$ if $m = 0$. In a nutshell,

$$F'_{1p} = F'_{2p} = \begin{cases} \dfrac{1}{2}\dfrac{{}_p\mathcal{C}_q^{-1}(|z|^2;m)}{{}_p\rho_q(m-1)} & \text{if} \quad m \geqslant 1 \\ 0 & \text{if} \quad m = 0. \end{cases} \tag{65}$$

Therefore, there is no squeezing either in X_{1u} or X_{2u} for the states $|p;q;z,+m\rangle$. Moreover, we have

$$\langle\Delta X_{1u}^2\rangle = \langle\Delta X_{2u}^2\rangle = \begin{cases} \frac{1}{2}|\langle[X_{1u}, X_{2u}]\rangle_{(+m)}| + \frac{1}{2}\frac{{}_p\mathcal{C}_q^{-1}(|z|^2;m)}{{}_p\rho_q(m-1)} & \text{if} \quad m \geqslant 1 \\ \frac{1}{2}|\langle[X_{1u}, X_{2u}]\rangle_{(0)}| & \text{if} \quad m = 0 \end{cases} \tag{66}$$

which indicates that, while the states $|p;q;z\rangle$ are intelligent states for the deformed operators X_{1u}, X_{2u}, the photon-added $|p;q;z,+m\rangle (m \geqslant 1)$ are not so. On the other hand, by using (58*a*)–(58*d*) we find that the states $|p;q;z,-m\rangle$ exhibit squeezing in the X_{1u} quadrature provided

$$\begin{aligned} F'_{1n} &= \left({}_p\tilde{S}'_q(|z|^2;m)\,{}_pU_q(|z|^2;m) - {}_pT_q^2(|z|^2;m)\right)\cos(2\theta) \\ &\quad + {}_p\tilde{S}'_q(|z|^2;m)\,{}_pV_q(|z|^2;m) - {}_pT_q^2(|z|^2;m) < 0 \end{aligned} \tag{67}$$

and that, in the quadrature X_{2u}, it occurs whenever

$$\begin{aligned} F'_{2n} &= \left({}_pT_q^2(|z|^2;m) - {}_p\tilde{S}'_q(|z|^2;m)\,{}_pU_q(|z|^2;m)\right)\cos(2\theta) \\ &\quad + {}_p\tilde{S}'_q(|z|^2;m)\,{}_pV_q(|z|^2;m) - {}_pT_q^2(|z|^2;m) < 0 \end{aligned} \tag{68}$$

where

$${}_pT_q(|z|^2;m) = \sum_{j=0}^{+\infty} \frac{|z|^{2j}\,{}_p\rho_q(j+1)}{{}_p\rho_q(j+m)\,{}_p\rho_q(j+m+1)} \tag{69a}$$

$${}_pU_q(|z|^2;m) = \sum_{j=0}^{+\infty} \frac{|z|^{2j}\,{}_p\rho_q(j+2)}{{}_p\rho_q(j+m)\,{}_p\rho_q(j+m+2)} \tag{69b}$$

$${}_pV_q(|z|^2;m) = \sum_{j=0}^{+\infty} \frac{|z|^{2(j-1)}\,{}_p\rho_q(j)\,{}_pf_q^2(j-1)}{({}_p\rho_q(j+m))^2}. \tag{69c}$$

In figure 4(*a*) and (*b*), respectively, we have shown the three-dimensional plots of $H'_{1n} \equiv 10^{-6}F'_{1n}$ (where F'_{1n} corresponds to squeezing of X_{1u} for the states $|-;b;z,-m\rangle$) and $H'_{2n} \equiv 10^{-6}F'_{2n}$ (where F'_{2n} corresponds to squeezing of X_{2u} for the states $|-;b;z,-m\rangle$) versus $x = |z|^2$ and θ for $b = 1/30$ and $m = 1$. It is seen that the quadrature operators X_{1u} (resp. X_{2u}) exhibit maximum squeezing at $\theta = 0$ (resp. $\theta = \pm\pi/2$). We observe that for $\theta = 0$ the fluctuation of X_{1u} is always unsqueezed while that of X_{2u} is squeezed.

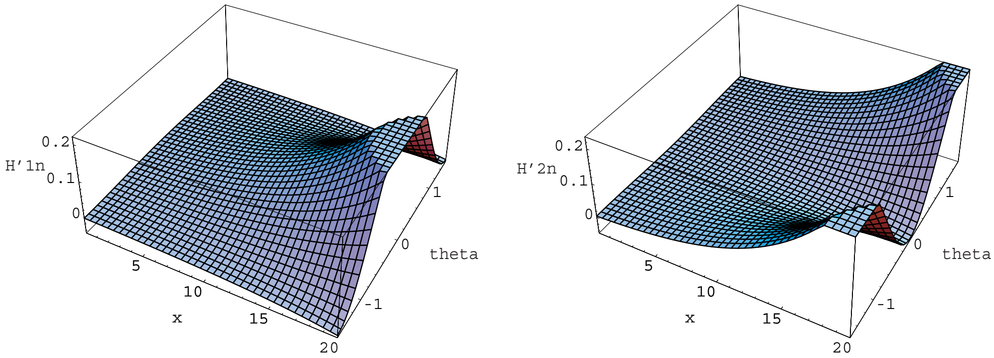
6.2. Photon number distribution in deformed quantum optics

In order to examine photon-counting statistics of deformed photons, one can generalize the notion of the Mandel parameter as follows:

$$Q_{h,\pm m} := \frac{\left\langle\left({}_pU_q^\dagger\right)^2{}_pU_q^2\right\rangle - \left(\left\langle{}_pU_q^\dagger\,{}_pU_q\right\rangle\right)^2}{\left\langle{}_pU_q^\dagger\,{}_pU_q\right\rangle}. \tag{70}$$

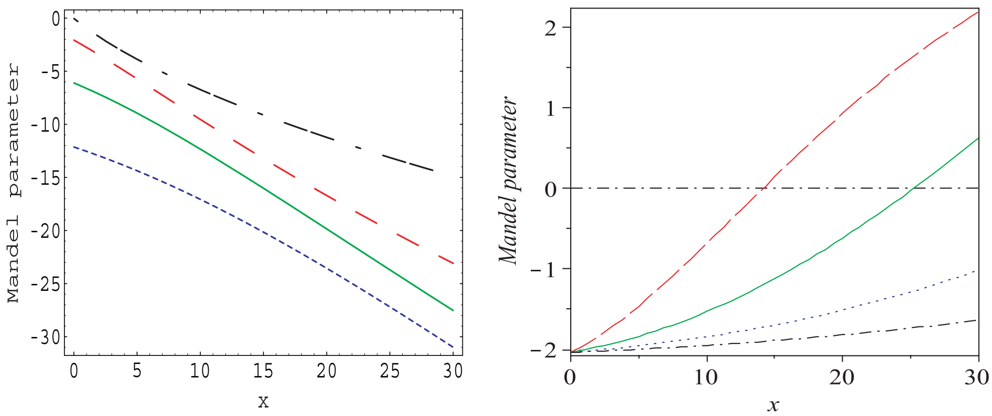
By using the above results, it is readily obtained

$$\begin{aligned} Q_{h,+m} &= \left(\sum_{j=0}^{+\infty}\frac{|z|^{2j}}{{}_p\rho_q(j+m-2)}\right)\left(\sum_{j=0}^{+\infty}\frac{|z|^{2j}}{{}_p\rho_q(j+m-1)}\right)^{-1} \\ &\quad - \left(\sum_{j=0}^{+\infty}\frac{|z|^{2j}}{{}_p\rho_q(j+m-1)}\right)\left(\sum_{j=0}^{+\infty}\frac{|z|^{2j}}{{}_p\rho_q(j+m)}\right)^{-1} \end{aligned} \tag{71}$$



(*a*) (*b*)

Figure 4. (*a*) Three-dimensional plot of $H'_{1n} \equiv 10^{-6} F'_{1n}$, corresponding to squeezing of X_1 for the states $|-; b; z, -m\rangle$ versus $x = |z|^2$ and θ for $b = 1/30, m = 1$. (*b*) Three-dimensional plot of $H'_{2n} \equiv 10^{-6} F'_{2n}$, corresponding to squeezing of X_2 for the states $|-; b; z, -m\rangle$ versus $x = |z|^2$ and θ for $b = 1/30, m = 1$.



(*a*) (*b*)

Figure 5. (*a*) Variation of the Mandel parameter Q_h for the states $|-; b; z, +m\rangle$ versus $x = |z|^2$: $b = \frac{1}{30}, m = 1$(dotdash); $b = \frac{1}{30}, m = 2$(dash); $b = \frac{1}{30}, m = 3$(solid); $b = \frac{1}{30}, m = 4$(dot). (*b*) Variation of the Mandel parameter Q_h for the states $|-; b; z, -m\rangle$ versus $x = |z|^2$:$b = \frac{1}{30}$, $m = 1$(dotdash); $b = \frac{1}{30}, m = 2$(dash); $b = \frac{1}{30}, m = 3$(solid); $b = \frac{1}{30}, m = 4$(dot).

for the states $|p; q; z, +m\rangle$ and

$$Q_{h,-m} = \left(\sum_{j=2}^{+\infty} \frac{|z|^{2j}\, {}_p\rho_q(j)\, {}_pf_q^2(j-1)\, {}_pf_q^2(j-2)}{({}_p\rho_q(j+m))^2}\right)\left(\sum_{j=1}^{+\infty} \frac{|z|^{2j}\, {}_p\rho_q(j)\, {}_pf_q^2(j-1)}{({}_p\rho_q(j+m))^2}\right)^{-1} - \left(\sum_{j=1}^{+\infty} \frac{|z|^{2j}\, {}_p\rho_q(j)\, {}_pf_q^2(j-1)}{({}_p\rho_q(j+m))^2}\right)\left(\sum_{j=0}^{+\infty} \frac{|z|^{2j}\, {}_p\rho_q(j)}{({}_p\rho_q(j+m))^2}\right)^{-1} \tag{72}$$

for the states $|p; q; z, -m\rangle$.

In figures 5(*a*) and (*b*), we show the effects of the parameters a, b and m on the variation of the Mandel parameter as a function of x for the states $|-; b; z, +m\rangle$ and $|-; b; z, -m\rangle$, respectively. We conclude from these numerical results that the states $|-; b; z, +m\rangle$ always exhibit sub-Poissonian statistics. It is worth noticing that in this case, for fixed b, the sub-Poissonian behavior is more pronounced as x and m increase, while as x increases, the states $|-; b; z, -m\rangle$ show a transition from sub-Poissonian to super-Poissonian statistics.

7. Concluding remarks

In the present paper, by making use of the properties of the inverse of the ladder operators ${}_pU_q$ and ${}_pU_q^\dagger$ in the Fock space, we have constructed photon-added and photon-depleted coherent states corresponding to the generalized hypergeometric coherent states $|p; q; z\rangle$. These states are introduced as the eigenstates of the combination of the ladder operators ${}_pU_q$ and ${}_pU_q^\dagger$ and their inverse. The completeness of the sets of the states $|p; q; z, +m\rangle$ and $|p; q; z, -m\rangle$, along with the number states $\{|j\rangle, j = 0, 1, 2, \ldots, m-1\}$, has been thoroughly discussed. In addition to the mathematical properties, the physical characteristics of the particular states $|-; b; z, +m\rangle$ and $|-; b; z, -m\rangle$ have been analytically and numerically investigated in the context of both nondeformed and deformed quantum optics. More specifically,

(i) it has been found that for conventional photons, described by the operators a and $a^\dagger$, the quadrature operator X_1 (resp. X_2) exhibits maximum squeezing at $\theta = \pm\pi/2$ (resp. $\theta = 0$) in the states $|-; b; z, +m\rangle$. At the states $|-; b; z, -m\rangle$, the quadrature operator X_1 (resp. X_2) exhibits maximum squeezing at $\theta = 0$ (resp. $\theta = \pm\pi/2$).

(ii) In the context of nondeformed quantum optics, the states $|-; b; z, +m\rangle$ show a sub-Poissonian statistics while the states $|-; b; z, -m\rangle$ are always super-Poissonian.

(iii) In the context of deformed photons described by ${}_pU_q$ and ${}_pU_q^\dagger$, the quadrature operator X_{1u} (resp. X_{2u}) shows maximum squeezing at $\theta = 0$ (resp. $\theta = \pm\pi/2$) in the states $|-; b; z, -m\rangle$. They are unsqueezed in the states $|p; q; z, +m\rangle$ for $m \geqslant 1$. Moreover, the states $|p; q; z, 0\rangle \equiv |p; q; z\rangle$ are intelligent in the quadrature operators X_{1u} and X_{2u}.

(iv) In the context of deformed quantum optics, the states $|-; b; z, +m\rangle$ show a sub-Poissonian statistics while the states $|-; b; z, -m\rangle$ show a transition from sub-Poissonian to super-Poissonian behavior.

Acknowledgments

This work is partially supported by the Abdus Salam International Centre for Theoretical Physics (ICTP, Trieste, Italy) through the OEA-ICMPA-Prj-15. The ICMPA is in partnership with the Daniel Iagolnitzer Foundation (DIF), France.

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